Kronecker's double series and exact asymptotic expansions for free models of statistical mechanics on torus

2002 J. Phys. A: Math. Gen. 35 5543

(http://iopscience.iop.org/0305-4470/35/27/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 5543–5561 PII: S0305-4470(02)36947-6

Kronecker's double series and exact asymptotic expansions for free models of statistical mechanics on torus

E V Ivashkevich[1], **N Sh Izmailian**[2,3] **and Chin-Kun Hu**[2]

[1] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna 141980, Russia
[2] Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
[3] Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan 375036, Armenia

E-mail: huck@phys.sinica.edu.tw

Received 9 May 2002
Published 28 June 2002
Online at stacks.iop.org/JPhysA/35/5543

Abstract
For the free models of statistical mechanics on torus, including the Ising model, dimer model and Gaussian model, exact asymptotic expansions of free energy, internal energy and specific heat in the vicinity of the critical point are found. It is shown that there is a direct relation between the terms of the expansion and Kronecker's double series. The latter can be expressed in terms of the elliptic θ-functions in all orders of the asymptotic expansion.

PACS numbers: 05.50.+q, 05.70.Jk, 64.60.Cn

1. Introduction

It is well known that the singularities in thermodynamic functions associated with a critical point occur only in the thermodynamic limit when dimension L of the system under consideration tends to infinity. In such a limit, the critical fluctuations are correlated over a distance of the order of correlation length ξ_{bulk} that may be defined as the length scale governing the exponential decay of correlation functions. Besides these two fundamental lengths, L and ξ_{bulk}, there is also the microscopic length of interactions a. Thermodynamic quantities thus may in principle depend on the dimensionless ratios ξ_{bulk}/L and a/L. The finite-size scaling (FSS) hypothesis [1] assumes that in the scaling interval, for temperatures so close to the critical point that $a \ll \xi_{\text{bulk}} \sim L$, the microscopic length drops and the behaviour of any thermodynamic quantity can be described in terms of the universal scaling function of the scaling variable $t = \xi_{\text{bulk}}/L$. However, non-universal corrections to FSS do exist. These can sometimes be viewed as asymptotic series in powers of a/L.

Two-dimensional models of statistical mechanics have long served as a proving ground in attempts to understand critical behaviour and to test the general ideas of FSS. Very few of them have been solved exactly [2]; notably, the so-called free models that can be treated in terms of non-interacting quasi-particles on the lattice. The Ising model [3] (which is equivalent to the model of free lattice fermions) and the Gaussian model (which can be viewed as the model of free lattice bosons) are the most prominent examples.

For all free models, the exact asymptotic expansion of the free energy on an infinite cylinder of circumference L can easily be obtained by direct application of the Euler–Maclaurin summation formula [4, 5]. However, derivation of such an expansion on a torus of area S and aspect ratio ρ is much more difficult problem. For the Ising model on torus, such an expansion was first studied by Ferdinand and Fisher [6]. Starting with the explicit expression for the partition function [7], they calculated two leading terms, f_{bulk} and $f_0(\rho)$, of the expansion

$$F_{T=T_c}(\rho, S) = f_{\text{bulk}}S + f_0(\rho) + \sum_{p=1}^{\infty} f_p(\rho)S^{-p}. \tag{1}$$

Their calculations have recently been taken further to get two next sub-leading terms [8–10].

The purpose of this paper is to derive *all* terms of the exact asymptotic expansion of the logarithm of the partition function on torus for a class of free exactly solvable models of statistical mechanics. Our approach is based on an intimate relation between the terms of the asymptotic expansion and the so-called Kronecker's double series. Besides the aesthetic appeal of the exact expansion, there is also physical motivation to study non-universal corrections to FSS. The problem is that in numerical simulations of lattice models one usually studies relatively small lattices. Therefore, to compare the results of high precision numerical simulations with the theoretical predictions one cannot neglect sub-leading corrections to FSS [11]. Non-universal terms in the asymptotic expansion also provide important information about the structure of irrelevant operators in conformal field theory [12].

2. Free models of statistical mechanics

In this section, we formulate three basic models of statistical mechanics: Ising model, dimer model and Gaussian model. These models are often referred to as 'free' since they were shown to be equivalent to free fermions or free bosons on the lattice. Partition functions of all these models can be written in terms of the only object—the partition function with twisted boundary conditions $Z_{\alpha,\beta}(\mu)$. Exact asymptotic expansion of $Z_{\alpha,\beta}(\mu)$ will be our main objective in the subsequent section.

2.1. Ising model

The Ising model is usually formulated as follows. Consider a planar square lattice of size $M \times N$ with periodic boundary conditions, i.e. torus. To each site (m, n) of the torus a spin variable is ascribed, s_{mn}, with two possible values: +1 or −1. Two nearest neighbour spins, say s_{mn} and s_{mn+1}, contribute a term $-J s_{mn}s_{mn+1}$ to the Hamiltonian, where J is some fixed energy. Therefore, the Hamiltonian is simply the sum of all such terms, one for each edge of the lattice:

$$H(s) = -J \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} (s_{mn}s_{m+1n} + s_{mn}s_{mn+1}). \tag{2}$$

The partition function of the Ising model is given by the sum over all spin configurations on the lattice:

$$Z_{\text{Ising}}(J) = \sum_{\{s\}} \mathrm{e}^{-H(s)}.$$

It is convenient to set up another parameterizations of the interaction constant J in terms of the mass variable $\mu = \ln\sqrt{\sinh 2J}$. Critical point corresponds to the massless case $\mu = 0$.

An explicit expression for the partition function of the Ising model on $M \times N$ torus, which was given originally by Kaufmann [13], can be written as

$$Z_{\text{Ising}}(\mu) = \tfrac{1}{2}(\sqrt{2}\,\mathrm{e}^{\mu})^{MN}\left\{Z_{\frac{1}{2},\frac{1}{2}}(\mu) + Z_{0,\frac{1}{2}}(\mu) + Z_{\frac{1}{2},0}(\mu) + Z_{0,0}(\mu)\right\} \tag{3}$$

where we have introduced the partition function with twisted boundary conditions

$$Z^2_{\alpha,\beta}(\mu) = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} 4\left[\sin^2\left(\frac{\pi(n+\alpha)}{N}\right) + \sin^2\left(\frac{\pi(m+\beta)}{M}\right) + 2\sinh^2\mu\right].$$

Here $\alpha = 0$ corresponds to the periodic boundary conditions for the underlying free fermion in the N-direction while $\alpha = \frac{1}{2}$ stands for anti-periodic boundary conditions. Similarly β controls boundary conditions in M-direction. With the help of the identity [14]

$$4|\sinh(M\omega + \mathrm{i}\pi\beta)|^2 = 4[\sinh^2 M\omega + \sin^2\pi\beta] = \prod_{m=0}^{M-1} 4\left[\sinh^2\omega + \sin^2\left(\frac{\pi(m+\beta)}{M}\right)\right]$$

the partition function with twisted boundary conditions $Z_{\alpha,\beta}$ can be transformed into a simpler form

$$Z_{\alpha,\beta}(\mu) = \prod_{n=0}^{N-1} 2\left|\sinh\left[M\omega_\mu\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]\right| \tag{4}$$

where lattice dispersion relation is

$$\omega_\mu(k) = \operatorname{arcsinh}\sqrt{\sin^2 k + 2\sinh^2\mu}. \tag{5}$$

This is nothing but the functional relation between energy ω_μ and momentum k of a free quasi-particle on the planar square lattice.

2.2. Dimer model

A 'dimer' is a two-atom molecule. When drawn on a lattice it covers two adjacent sites of the lattice and the bond that joins them. The 'dimer problem' is to determine the number of ways of covering a given lattice with dimers, so that all sites are occupied and no two dimers overlap. If we consider planar square lattice of size $2M \times 2N$ wrapped on a torus then the number of dimers must be $2MN$ and the number of distinct dense coverings of the lattice (the partition function) has been calculated by Kasteleyn and Fisher [15, 16]. This can be expressed in terms of the same partition function with twisted boundary conditions (4) as

$$Z_{\text{Dimer}} = \tfrac{1}{2}\left\{Z^2_{\frac{1}{2},\frac{1}{2}}(0) + Z^2_{0,\frac{1}{2}}(0) + Z^2_{\frac{1}{2},0}(0) - Z^2_{0,0}(0)\right\}. \tag{6}$$

First two leading terms of the asymptotic expansion of this partition function have been obtained by Ferdinand [17]. Let us also mention that dimers are always in the critical point and have no phase transition.

2.3. Gaussian model

Let us now turn to a boson analogue of Ising model, which is often referred to as Gaussian model. Again, we consider planar square lattice of size $N \times M$ wrapped on a torus. To each site (m, n) of the lattice we assign a continuous variable x_{mn}. The Hamiltonian of the model is

$$H(x) = -J\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left(x_{mn}x_{m+1n} + x_{mn}x_{mn+1} - 2x^2_{mn}\right). \tag{7}$$

The partition function of the model can be written as

$$Z(J) = \int_{\mathbf{R}^{MN}} \mathrm{e}^{-H(x)}\, \mathrm{d}\sigma(x).$$

If the measure $\mathrm{d}\sigma(x)$ in the phase space $\mathbf{R}^{MN}$ is chosen to be Gaussian,

$$\mathrm{d}\sigma_{\mathrm{Gauss}}(x) = \pi^{-MN/2} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} \mathrm{e}^{-x_{mn}^2}\, \mathrm{d}x_{mn}$$

the integration can be done explicitly and the partition function of the free boson model can be written in terms of the partition function with twisted boundary conditions (4) and parameterization $J^{-1} = 4\cosh^2\mu$ as

$$Z_{\mathrm{Gauss}}(\mu) = (\sqrt{2}\cosh\mu)^{MN} [Z_{0,0}(\mu)]^{-1}. \tag{8}$$

This model exhibits phase transition at the point $\mu_c = 0$ where the partition function is divergent. This is due to the presence of the so-called zero mode, i.e. due to the symmetry transformation $x_{mn} \to x_{mn} + \mathrm{const}$, which leaves Hamiltonian (7) invariant. Correlation functions of disorder operator in this model have been studied by Sato *et al* [18].

The reason why this model is often considered as boson analogue of the Ising model is that one can choose another measure in the phase space, which makes this model equivalent to the Ising model considered above,

$$\mathrm{d}\sigma_{\mathrm{Ising}}(x) = 2^{-MN} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} [\delta(x_{mn} - 1) + \delta(x_{mn} + 1)]\, \mathrm{d}x_{mn}$$

where δ are Dirac δ-functions. With such a definition, the variables x_{mn} can actually take only two values: +1 or -1, so that $x_{mn}^2 = 1$. In this case, integration can be replaced by summation over discrete values of $x_{mn} = \pm 1$ and Hamiltonian (7) coincides with the Hamiltonian of the Ising model (2) up to a constant.

3. Asymptotic expansion in terms of Kronecker's double series

In the previous section, it was shown that partition functions of three basic free models of statistical mechanics can be expressed in terms of the only object—the partition function with twisted boundary condition $Z_{\alpha,\beta}(\mu)$. In this section, we shall obtain exact asymptotic $1/S$-expansion of the partition function near the critical point. For reader's convenience, all the technical details of our calculations and the definitions of the special functions are summarized in the appendices attached to the paper.

First of all, let us mention the symmetry properties of the partition function $Z_{\alpha,\beta}(\mu)$. From its definition (4) one can easily verify that it is even and periodic with respect to its arguments α and β:

$$Z_{\alpha,\beta}(\mu) = Z_{\alpha,-\beta}(\mu) = Z_{-\alpha,\beta}(\mu)$$

$$Z_{\alpha,\beta}(\mu) = Z_{1+\alpha,\beta}(\mu) = Z_{\alpha,1+\beta}(\mu).$$

These imply that twist angles α and β can be taken from the interval [0, 1]. Then, one can note that for all twists $(\alpha, \beta) \neq (0, 0)$ the partition function $Z_{\alpha,\beta}(\mu)$ is even with respect to its mass argument μ. Hence, near the critical point ($\mu = 0$) we have

$$Z_{\alpha,\beta}(\mu) = Z_{\alpha,\beta}(0) + \frac{\mu^2}{2!} Z''_{\alpha,\beta}(0) + \cdots \qquad (\alpha, \beta) \neq (0, 0). \tag{9}$$

The only exception is the point where both α and β are equal to zero. This case has to be treated separately since at this point the partition function turns to zero. As a result, we have

$$Z_{0,0}(\mu) = \mu Z'_{0,0}(0) + \frac{\mu^3}{3!} Z'''_{0,0}(0) + \cdots \qquad (\alpha, \beta) = (0, 0). \tag{10}$$

In what follows, notation $Z_{\alpha,\beta}(\mu)$ will imply $(\alpha, \beta) \neq (0, 0)$.

3.1. Asymptotic expansion of $Z_{\alpha,\beta}(0)$

Considering the logarithm of the partition function with twisted boundary conditions (4), we note that it can be transformed as

$$\ln Z_{\alpha,\beta}(0) = M \sum_{n=0}^{N-1} \omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \sum_{n=0}^{N-1} \ln\left|1 - \exp\left\{-2\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]\right\}\right|. \tag{11}$$

The second sum here vanishes in the formal limit $M \to \infty$ when the torus turns into infinitely long cylinder of circumference N. Therefore, the first sum gives the logarithm of the partition function with twisted angle α on that cylinder. Its asymptotic expansion can be found with the help of the Euler–Maclaurin summation formula (appendix A)

$$M \sum_{n=0}^{N-1} \omega_0\left(\frac{\pi(n+\alpha)}{N}\right) = \frac{S}{\pi}\int_0^{\pi} \omega_0(x)\,\mathrm{d}x - \pi\lambda\rho B_2^{\alpha} - 2\pi\rho \sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^p \frac{\lambda_{2p}}{(2p)!}\frac{B_{2p+2}^{\alpha}}{2p+2} \tag{12}$$

where $\int_0^{\pi}\omega_0(x)\,\mathrm{d}x = 2\gamma$, $\gamma = 0.915\,965...$ is Catalan's constant and B_p^{α} are so-called Bernoulli polynomials. We have also used the symmetry property, $\omega_0(k) = \omega_0(\pi - k)$, of the lattice dispersion relation (5) and its Taylor expansion

$$\omega_0(k) = k\left(\lambda + \sum_{p=1}^{\infty} \frac{\lambda_{2p}}{(2p)!} k^{2p}\right) \tag{13}$$

where $\lambda = 1$, $\lambda_2 = -2/3$, $\lambda_4 = 4$, etc. In what follows, we shall not use the special values of these coefficients assuming the possibility for generalizations.

We may transform the second term in equation (11) as

$$\begin{aligned}
&\sum_{n=0}^{N-1} \ln\left|1 - \exp\left\{-2\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]\right\}\right| \\
&\quad = -\,\mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\sum_{n=0}^{[N/2]-1}\exp\left\{-2m\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]\right\}\right. \\
&\qquad \left. + \sum_{n=0}^{N-[N/2]-1}\exp\left\{-2m\left[M\omega_0\left(\frac{\pi(n+1-\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]\right\}\right\}.
\end{aligned} \tag{14}$$

The argument of the first exponent can be expanded in powers of $1/S$ if we replace the lattice dispersion relation $\omega_0(x)$ with its Taylor expansion (13):

$$\exp\left\{-2\pi m[\lambda\rho(n+\alpha) + \mathrm{i}\beta] - 2\pi m\rho\sum_{p=1}^{\infty}\frac{\lambda_{2p}}{(2p)!}\left(\frac{\pi^2\rho}{S}\right)^p (n+\alpha)^{2p+1}\right\}.$$

Taking into account the relation between moments and cumulants (appendix B), we obtain asymptotic expansion of the first exponent itself in powers of $1/S$,

$$\exp\left\{-2m\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right)+\mathrm{i}\pi\beta\right]\right\} = \exp\{-2\pi m[\lambda\rho(n+\alpha)+\mathrm{i}\beta]\}$$

$$-2\pi m\rho\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}(n+\alpha)^{2p+1}\exp\{-2\pi m[\lambda\rho(n+\alpha)+\mathrm{i}\beta]\}.$$

The differential operators Λ_{2p} that have appeared here can be expressed via coefficients λ_{2p} of the expansion of the lattice dispersion relation (13) as

$$\Lambda_2 = \lambda_2$$

$$\Lambda_4 = \lambda_4 + 3\lambda_2^2\frac{\partial}{\partial\lambda}$$

$$\Lambda_6 = \lambda_6 + 15\lambda_4\lambda_2\frac{\partial}{\partial\lambda} + 15\lambda_2^3\frac{\partial^2}{\partial\lambda^2}$$

$$\vdots$$

$$\Lambda_p = \sum_{r=1}^{p}\sum\left(\frac{\lambda_{p_1}}{p_1!}\right)^{k_1}\ldots\left(\frac{\lambda_{p_r}}{p_r!}\right)^{k_r}\frac{p!}{k_1!\ldots k_r!}\frac{\partial^k}{\partial\lambda^k}.$$

Here summation is over all positive numbers $\{k_1,\ldots,k_r\}$ and different positive numbers $\{p_1,\ldots,p_r\}$ such that $p_1k_1+\cdots+p_rk_r = p$ and $k = k_1+\cdots+k_r-1$.

The expansion for the second exponent in equation (14) can be obtained along the same lines by substitution: $\alpha\to 1-\alpha$. Plugging the expansion of both of the exponents back into equation (14) we obtain

$$\sum_{n=0}^{N-1}\ln\left|1-\exp\left\{-2\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right)+\mathrm{i}\pi\beta\right]\right\}\right|$$

$$= -\mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\sum_{n=0}^{[N/2]-1}\exp\{-2\pi m[\lambda\rho(n+\alpha)+\mathrm{i}\beta]\}\right.$$

$$\left.+\sum_{n=0}^{N-[N/2]-1}\exp\{-2\pi m[\lambda\rho(n+1-\alpha)+\mathrm{i}\beta]\}\right\}$$

$$+2\pi\rho\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}\mathrm{Re}\sum_{m=1}^{\infty}\left\{\sum_{n=0}^{[N/2]-1}(n+\alpha)^{2p+1}\right.$$

$$\times\exp\{-2\pi m[\lambda\rho(n+\alpha)+\mathrm{i}\beta]\}$$

$$\left.+\sum_{n=0}^{N-[N/2]-1}(n+1-\alpha)^{2p+1}\exp\{-2\pi m[\lambda\rho(n+1-\alpha)+\mathrm{i}\beta]\}\right\}.$$

In all these series, summation over n can be extended to infinity. The resulting errors are exponentially small and do not affect our asymptotic expansion in any finite power of $1/S$.

The key point of our analysis is the observation that all the series that have appeared in such an expansion can be obtained by resummation of either elliptic theta function $\theta_{\alpha,\beta}(\tau)$ (appendix C) or Kronecker's double series $K_p^{\alpha,\beta}(\tau)$ (appendix D). Namely, with the help of identities (C.2) and (D.1) we obtain

$$\sum_{n=0}^{N-1} \ln \left| 1 - \exp\left\{ -2\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta \right]\right\}\right| = \ln\left|\frac{\theta_{\alpha,\beta}(\mathrm{i}\lambda\rho)}{\eta(\mathrm{i}\lambda\rho)}\right| + \pi\lambda\rho B_2^{\alpha}$$
$$- 2\pi\rho \sum_{p=1}^{\infty} \left(\frac{\pi^2\rho}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!} \frac{\mathrm{Re} K_{2p+2}^{\alpha,\beta}(\mathrm{i}\lambda\rho) - B_{2p+2}^{\alpha}}{2p+2}. \tag{15}$$

Substituting equations (12) and (15) into equation (11) we finally obtain exact asymptotic expansion of the logarithm of the partition function with twisted boundary conditions in terms of Kronecker's double series:

$$\ln Z_{\alpha,\beta}(0) = \frac{S}{\pi}\int_0^{\pi} \omega_0(x)\,\mathrm{d}x + \ln\left|\frac{\theta_{\alpha,\beta}(\mathrm{i}\lambda\rho)}{\eta(\mathrm{i}\lambda\rho)}\right| - 2\pi\rho\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{\mathrm{Re} K_{2p+2}^{\alpha,\beta}(\mathrm{i}\lambda\rho)}{2p+2}. \tag{16}$$

Note that Bernoulli polynomials B_p^{α} have finally dropped out of the asymptotic expansion on torus. This actually means that Kronecker's double series can be considered as elliptic generalizations of Bernoulli polynomials.

3.2. Asymptotic expansion of $Z'_{0,0}(0)$

As it has already been mentioned, we have to treat the case $(\alpha, \beta) = (0, 0)$ separately. Taking the derivative of equation (4) with respect to mass variable μ and then considering limit $\mu \to 0$ we obtain

$$Z'_{0,0}(0) = 2\sqrt{2}M \prod_{n=1}^{N-1} 2\left|\sinh\left[M\omega_0\left(\frac{\pi n}{N}\right)\right]\right|.$$

Asymptotic expansion of this expression can be found along the same lines as above. In terms of Kronecker's double series, the expansion can be written as

$$\ln Z'_{0,0}(0) = \frac{S}{\pi}\int_0^{\pi}\omega_0(x)\,\mathrm{d}x + \frac{1}{2}\ln 8\rho S + 2\ln|\eta(\mathrm{i}\lambda\rho)|$$
$$- 2\pi\rho\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}\frac{\mathrm{Re} K_{2p+2}^{0,0}(\mathrm{i}\lambda\rho)}{2p+2}. \tag{17}$$

3.3. Asymptotic expansion of $Z''_{\alpha,\beta}(0)$

The analysis of the $Z''_{\alpha,\beta}(0)$ is a little more involved. Taking the second derivative of equation (4) with respect to mass variable μ and then considering limit $\mu \to 0$ we obtain

$$\frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} = \mathrm{Re}\, M\sum_{n=0}^{N-1}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right)\coth\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]$$
$$= M\sum_{n=0}^{N-1}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right) + 2\,\mathrm{Re}\, M\sum_{n=0}^{N-1}\sum_{m=1}^{\infty}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right)$$
$$\times\exp\left\{-2m\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right) + \mathrm{i}\pi\beta\right]\right\} \tag{18}$$

where $\omega_0''(x)$ is the second derivative of $\omega_\mu(x)$ with respect to μ at criticality

$$\omega_0''(x) = \frac{2}{\sin x\sqrt{1+\sin^2 x}}.$$

Using Taylor's theorem, the asymptotic expansion of the $\omega_0''(x)$ can be written in the following form:

$$\omega_0''(x) = \frac{\kappa}{x}\left[1+\sum_{p=1}^{\infty}\frac{\kappa_{2p}}{(2p)!}x^{2p}\right]$$

where $\kappa = 2$, $\kappa_2 = -2/3$, $\kappa_4 = 172/15$, etc. Again, in what follows, we shall not use the special values of these coefficients assuming the possibility for generalizations.

We may transform the first sum in equation (18) as

$$M\sum_{n=0}^{N-1}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right) = M\sum_{n=0}^{N-1} f\left(\frac{\pi(n+\alpha)}{N}\right) + \frac{\kappa S}{\pi}\sum_{n=0}^{N-1}\left(\frac{1}{n+\alpha}+\frac{1}{n+1-\alpha}\right) \tag{19}$$

where we have introduced the function $f(x) = \omega_0''(x) - \kappa/x - \kappa/(\pi - x)$. This function and all its derivatives are integrable over the interval $(0, \pi)$. Thus, for the first term in equation (19) we may use again the Euler–Maclaurin summation formula (appendix A), and after a little algebra we obtain

$$M\sum_{n=0}^{N-1} f\left(\frac{\pi(n+\alpha)}{N}\right) = \frac{S}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x - \pi\rho\kappa\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^{p-1}\frac{\kappa_{2p}B_{2p}^{\alpha}}{p(2p)!} + \frac{\kappa S}{\pi}\sum_{p=1}^{\infty}\frac{B_{2p}^{\alpha}}{p}\frac{1}{N^{2p}} \tag{20}$$

where $\int_0^{\pi} f(x)\,\mathrm{d}x = 2\ln 2 - 4\ln\pi$. The second sum in equation (19) can be written in terms of the digamma function $\psi(x)$:

$$\sum_{n=0}^{N-1}\left(\frac{1}{n+\alpha}+\frac{1}{n+1-\alpha}\right) = [\psi(N+\alpha)+\psi(N+1-\alpha)-\psi(\alpha)-\psi(1-\alpha)]. \tag{21}$$

The asymptotic expansion of the digamma function $\psi(x)$ is given by (see appendix E)

$$\psi(N+\alpha) = \ln N - \sum_{p=1}^{\infty}(-1)^p\frac{B_p^{\alpha}}{p}\frac{1}{N^p}. \tag{22}$$

Using the symmetry properties of the Bernoulli polynomials B_p^{α}, namely $B_{2p}^{1-\alpha} = B_{2p}^{\alpha}$ and $B_{2p+1}^{1-\alpha} = -B_{2p+1}^{\alpha}$, equation (21) can be rewritten as

$$\sum_{n=0}^{N-1}\left(\frac{1}{n+\alpha}+\frac{1}{n+1-\alpha}\right) = 2\ln N - \sum_{p=1}^{\infty}\frac{B_{2p}^{\alpha}}{p}\frac{1}{N^{2p}} - \psi(\alpha) - \psi(1-\alpha). \tag{23}$$

Plugging equations (20) and (23) back in equation (19) we finally obtain

$$M\sum_{n=0}^{N-1}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right) = \frac{2\kappa S}{\pi}\left[\frac{1}{2\kappa}\int_0^{\pi} f(x)\,\mathrm{d}x + \ln N - \frac{\psi(\alpha)+\psi(1-\alpha)}{2}\right]$$
$$- \pi\rho\kappa\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^{p-1}\frac{\kappa_{2p}B_{2p}^{\alpha}}{p(2p)!}. \tag{24}$$

Let us now consider the second sum in equation (18). Note that function $\omega_0''(x)$ can be represented as

$$\omega_0''(x) = \frac{\kappa}{x}\exp\left\{\sum_{p=1}^{\infty}\frac{\varepsilon_{2p}}{(2p)!}x^{2p}\right\} \tag{25}$$

where coefficients ε_{2p} and κ_{2p} are related to each other through relation between moments and cumulants (appendix B). Following along the same lines as in section 3.1, the second sum in equation (18) can be written as

$$\begin{aligned}
2\,\mathrm{Re}\,M\sum_{n=0}^{N-1}\sum_{m=1}^{\infty}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right)\exp\left\{-2m\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right)+\mathrm{i}\pi\beta\right]\right\}&\\
= \frac{2\kappa S}{\pi}\mathrm{Re}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{n+\alpha}\exp\{-2\pi m[\rho(n+\alpha)+\mathrm{i}\beta]\}\right.&\\
\left.+\frac{1}{n+1-\alpha}\exp\{-2\pi m[\rho(n+1-\alpha)+\mathrm{i}\beta]\}\right\}&\\
+\kappa\pi\rho\Omega_2\,\mathrm{Re}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\{(n+\alpha)\exp\{-2\pi m[\lambda\rho(n+\alpha)+\mathrm{i}\beta]\}&\\
+(n+1-\alpha)\exp\{-2\pi m[\lambda\rho(n+1-\alpha)+\mathrm{i}\beta]\}\}&\\
-\kappa\pi\rho\sum_{p=2}^{\infty}\frac{\Omega_{2p}}{p(2p)!}\left(\frac{\pi^2\rho}{S}\right)^{p-1}\mathrm{Re}\,K_{2p}^{\alpha,\beta}(\mathrm{i}\lambda\rho)+\kappa\pi\rho\sum_{p=2}^{\infty}\frac{\kappa_{2p}B_{2p}^{\alpha}}{p(2p)!}\left(\frac{\pi^2\rho}{S}\right)^{p-1}.&
\end{aligned} \tag{26}$$

The differential operators Ω_{2p} that have appeared here can be expressed via coefficients $\omega_{2p} = \varepsilon_{2p} + \lambda_{2p}\frac{\partial}{\partial\lambda}$ as

$$\begin{aligned}
\Omega_2 &= \omega_2\\
\Omega_4 &= \omega_4 + 3\omega_2^2\\
&\vdots
\end{aligned}$$

Let us introduce the function $R_{\alpha,\beta}(\rho)$,

$$\begin{aligned}
R_{\alpha,\beta}(\rho) = -\frac{\psi(\alpha)+\psi(1-\alpha)}{2} + \mathrm{Re}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{n+\alpha}\exp\{-2\pi m[\rho(n+\alpha)+\mathrm{i}\beta]\}\right.&\\
\left.+\frac{1}{n+1-\alpha}\exp\{-2\pi m[\rho(n+1-\alpha)+\mathrm{i}\beta]\}\right\}&
\end{aligned} \tag{27}$$

whose first derivative with respect to ρ is given by (see appendix C, equation (C.4))

$$\frac{\partial}{\partial\rho}R_{\alpha,\beta}(\rho) = -2\frac{\partial}{\partial\rho}\ln|\theta_{\alpha,\beta}(\mathrm{i}\rho)| + \frac{1}{2\pi}\left(\frac{\partial}{\partial z}\ln|\theta_{\alpha,\beta}(\mathrm{i}\rho)|\right)^2. \tag{28}$$

For the cases $(\alpha,\beta) = (0, 1/2), (1/2, 0), (1/2, 1/2)$ the second term in above equation is equal to zero, and for $R_{\alpha,\beta}(\rho)$ we obtain

$$R_{\alpha,\beta}(\rho) = -2\ln|\theta_{\alpha,\beta}(\mathrm{i}\rho)| + C_E + 2\ln 2 \tag{29}$$

where C_E is the Euler constant.

With the help of identity (C.3) equation (26) can be rewritten as

$$2\,\mathrm{Re}\,M\sum_{n=0}^{N-1}\sum_{m=1}^{\infty}\omega_0''\left(\frac{\pi(n+\alpha)}{N}\right)\exp\left\{-2m\left[M\omega_0\left(\frac{\pi(n+\alpha)}{N}\right)+\mathrm{i}\pi\beta\right]\right\}$$
$$=\frac{2\kappa S}{\pi}\left[R_{\alpha,\beta}(\rho)+\frac{\psi(\alpha)+\psi(1-\alpha)}{2}\right]+\frac{\kappa}{2}\left(\kappa_2\rho\frac{\partial}{\partial\rho}+\lambda_2\rho^2\frac{\partial^2}{\partial\rho^2}\right)\ln\left|\frac{\theta_{\alpha,\beta}(\mathrm{i}\rho)}{\eta(\mathrm{i}\rho)}\right|$$
$$-\kappa\pi\rho\sum_{p=2}^{\infty}\frac{\Omega_{2p}}{p(2p)!}\left(\frac{\pi^2\rho}{S}\right)^{p-1}\mathrm{Re}\,K_{2p}^{\alpha,\beta}(\mathrm{i}\lambda\rho)+\kappa\pi\rho\sum_{p=1}^{\infty}\frac{\kappa_{2p}B_{2p}^{\alpha}}{p(2p)!}\left(\frac{\pi^2\rho}{S}\right)^{p-1}. \tag{30}$$

Substituting equations (24) and (30) into equation (18) we finally obtain exact asymptotic expansion of the $Z''_{\alpha,\beta}(0)$:

$$\frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)}=\frac{2\kappa S}{\pi}\left[\frac{1}{2\kappa}\int_0^{\pi}f(x)\,\mathrm{d}x+\ln N+R_{\alpha,\beta}(\rho)\right]+\frac{\kappa}{2}\left(\kappa_2\rho\frac{\partial}{\partial\rho}+\lambda_2\rho^2\frac{\partial^2}{\partial\rho^2}\right)\ln\left|\frac{\theta_{\alpha,\beta}(\mathrm{i}\rho)}{\eta(\mathrm{i}\rho)}\right|$$
$$-\kappa\pi\rho\sum_{p=2}^{\infty}\frac{\Omega_{2p}}{p(2p)!}\left(\frac{\pi^2\rho}{S}\right)^{p-1}\mathrm{Re}\,K_{2p}^{\alpha,\beta}(\mathrm{i}\lambda\rho). \tag{31}$$

3.4. Asymptotic expansion of $Z'''_{0,0}(0)$

Let us now consider the case $(\alpha,\beta)=(0,0)$. Equation (10) implies immediately that

$$\lim_{\mu\to 0}\frac{Z''_{0,0}(\mu)}{Z_{0,0}(\mu)}=\frac{Z'''_{0,0}(0)}{Z'_{0,0}(0)}. \tag{32}$$

Taking the third derivative of equation (4) with respect to mass variable μ and then considering limit $\mu\to 0$ we obtain

$$\frac{Z'''_{0,0}(0)}{Z'_{0,0}(0)}=2M^2-1+3M\sum_{n=1}^{N-1}\omega_0''\left(\frac{\pi n}{N}\right)\coth\left[M\omega_0\left(\frac{\pi n}{N}\right)\right]. \tag{33}$$

Asymptotic expansion of the $Z'''_{0,0}(0)$ can be found along the same lines as above. In terms of Kronecker's double series, the expansion can be written as

$$\frac{Z'''_{0,0}(0)}{Z'_{0,0}(0)}=\frac{6\kappa S}{\pi}\left[\frac{1}{2\kappa}\int_0^{\pi}f(x)\,\mathrm{d}x+\ln N+C_E-2\ln\eta(\mathrm{i}\rho)\right]$$
$$+3\kappa\left(\kappa_2\rho\frac{\partial}{\partial\rho}+\lambda_2\rho^2\frac{\partial^2}{\partial\rho^2}\right)\ln\eta(\mathrm{i}\rho)-1$$
$$-3\kappa\pi\rho\sum_{p=2}^{\infty}\frac{\Omega_{2p}}{p(2p)!}\left(\frac{\pi^2\rho}{S}\right)^{p-1}K_{2p}^{0,0}(\mathrm{i}\lambda\rho). \tag{34}$$

Expansions (16), (17), (31) and (34) are the main results of the paper. Kronecker's double series $K_p^{\alpha,\beta}$ with α and β taking values 0 and 1/2 can all be expressed in terms of the elliptic θ-functions only (appendix F).

4. Asymptotic expansion of the free energy, the internal energy and the specific heat

After reaching this point, one can easily write down all the terms of the exact asymptotic expansion (1) of the free energy, $F=-\ln Z$, at the critical point for all three models under consideration. For the Ising model we have found that the exact asymptotic expansion of the internal energy, $U=-\partial\ln Z(J)/\partial J$, and the specific heat, $C=\partial^2\ln Z(J)/\partial J^2$, at the critical point can be written in the following form:

$$U = u_{\text{bulk}} S + \sum_{p=0}^{\infty} u_p(\rho) S^{-p+\frac{1}{2}} \qquad C = c_{\text{bulk}} S + \sum_{p=0}^{\infty} c_{2p}(\rho) S^{-p+\frac{1}{2}} + \sum_{p=0}^{\infty} c_{2p+1}(\rho) S^{-p}.$$

And finally, the exact asymptotic expansion of the internal energy of the Gaussian model can be written in the following form

$$U = u_{\text{bulk}} S + \sum_{p=0}^{\infty} u_{2p+1}(\rho) S^{-p}.$$

(i) The asymptotic expansions of the free energy, the internal energy and the specific heat of the Ising model are

$$f_{\text{bulk}} = -\ln\sqrt{2} - \frac{2\gamma}{\pi}$$

$$f_0(\rho) = -\ln\frac{\theta_2+\theta_3+\theta_4}{2\eta}$$

$$f_1(\rho) = -\frac{\pi^3\rho^2}{180}\frac{\frac{7}{8}\left(\theta_2^9+\theta_3^9+\theta_4^9\right)+\theta_2\theta_3\theta_4\left[\theta_2^3\theta_4^3-\theta_3^3\theta_2^3-\theta_3^3\theta_4^3\right]}{\theta_2+\theta_3+\theta_4}$$

$$\begin{aligned}
f_2(\rho) = &-\frac{\pi^6\rho^5}{18\,432}\frac{\theta_2\theta_3\theta_4\left\{\theta_3^7\left(\theta_2^4-\theta_4^4\right)^2+\theta_2^7\left(\theta_3^4+\theta_4^4\right)^2+\theta_4^7\left(\theta_3^4+\theta_2^4\right)^2\right\}}{(\theta_2+\theta_3+\theta_4)^2}\\
&-\frac{31\pi^5\rho^4}{24\,192}\frac{\theta_3^4\left(\theta_4^9-\theta_2^9\right)+\theta_2^4\left(\theta_4^9-\theta_3^9\right)+\theta_4^4\left(\theta_3^9-\theta_2^9\right)}{\theta_2+\theta_3+\theta_4}\left(1+4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)\\
&-\frac{\pi^5\rho^4}{1512}\frac{\theta_2\theta_3\theta_4\left[\theta_3^3\left(\theta_2^7-\theta_4^7\right)+\theta_2^3\left(\theta_3^7-\theta_4^7\right)+\theta_4^3\left(\theta_2^7-\theta_3^7\right)\right]}{\theta_2+\theta_3+\theta_4}\left(1+4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)\\
&-\frac{31\pi^6\rho^5}{72\,576}\frac{\theta_3^4\theta_4^4\left[3\theta_3^4\theta_4^4\left(\theta_3+\theta_4\right)-\theta_2^4\left(2\theta_2^5+\theta_3^5-\theta_4^5\right)\right]}{\theta_2+\theta_3+\theta_4}\\
&-\frac{\pi^6\rho^5}{4536}\frac{\theta_3^4\theta_4^4\left[\theta_2\theta_3\left(\theta_2^7-\theta_3^7\right)+\theta_2\theta_4\left(\theta_2^7-\theta_4^7\right)+2\theta_2^4\theta_3\theta_4\left(\theta_3^3-\theta_4^3\right)-4\theta_2\theta_3^4\theta_4^4\right]}{\theta_2+\theta_3+\theta_4}
\end{aligned}$$

$$\vdots$$

$$u_{\text{bulk}} = -\sqrt{2}$$

$$u_0(\rho) = -2\sqrt{\rho}\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}$$

$$u_1(\rho) = \frac{\pi^3\rho^{5/2}}{48}\frac{\theta_2\theta_3\theta_4\left(\theta_2^9+\theta_3^9+\theta_4^9\right)}{(\theta_2+\theta_3+\theta_4)^2}$$

$$\begin{aligned}
u_2(\rho) = &\frac{\pi^6\rho^{9/2}}{4608}\frac{\theta_2^2\theta_3^2\theta_4^2\left[\theta_3^7\left(\theta_2^4-\theta_4^4\right)^2+\theta_2^7\left(\theta_3^4+\theta_4^4\right)^2+\theta_4^7\left(\theta_3^4+\theta_2^4\right)^2\right]}{(\theta_2+\theta_3+\theta_4)^3}\\
&+\frac{\pi^6\rho^{9/2}}{9216}\frac{\theta_2\theta_3\theta_4\left[23\left(\theta_2^{17}+\theta_3^{17}+\theta_4^{17}\right)+8\theta_2^4\theta_4^4\left(5\theta_2^9+5\theta_4^9-8\theta_3^9+2\theta_2^4\theta_4^5+2\theta_2^5\theta_4^4\right)\right]}{(\theta_2+\theta_3+\theta_4)^2}\\
&-\frac{\pi^5\rho^{7/2}}{192}\frac{\theta_2\theta_3\theta_4\left[\theta_3^9\left(\theta_2^4-\theta_4^4\right)+\theta_2^9\left(\theta_3^4+\theta_4^4\right)-\theta_4^9\left(\theta_3^4+\theta_2^4\right)\right]}{(\theta_2+\theta_3+\theta_4)^2}\left(1+\frac{\pi\rho}{2}\theta_3^4+4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)
\end{aligned}$$

$$\vdots$$

$$c_{\text{bulk}}(\rho) = \frac{8}{\pi}\left(\ln\sqrt{\frac{S}{\rho}}+\ln\frac{2^{5/2}}{\pi}+C_E-\frac{\pi}{4}\right)-4\rho\left(\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}\right)^2-\frac{16}{\pi}\frac{\sum_{i=2}^4\theta_i\ln\theta_i}{\theta_2+\theta_3+\theta_4}$$

$$c_0(\rho) = -2\sqrt{2}\sqrt{\rho}\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}$$

$$c_1(\rho) = \frac{\pi^2\rho^2}{6}\frac{\left(\theta_2^8-\theta_3^8\right)\theta_2\theta_3\ln\frac{\theta_3}{\theta_2}+\left(\theta_4^8-\theta_3^8\right)\theta_4\theta_3\ln\frac{\theta_3}{\theta_4}+\left(\theta_2^8-\theta_4^8\right)\theta_4\theta_2\ln\frac{\theta_4}{\theta_2}}{(\theta_2+\theta_3+\theta_4)^2}$$

$$+\frac{\pi^2\rho^2}{9}\frac{\theta_3^4\theta_4^4\left(2\theta_2-\theta_3-\theta_4\right)}{\theta_2+\theta_3+\theta_4}+\frac{\pi^3\rho^3}{12}\frac{\theta_2^2\theta_3^2\theta_4^2\left(\theta_2^9+\theta_3^9+\theta_4^9\right)}{(\theta_2+\theta_3+\theta_4)^3}$$

$$+\frac{\pi\rho}{9}\frac{\theta_2^5-\theta_4^5+\theta_3\left(\theta_2^4-\theta_4^4\right)-2\theta_2\theta_4\left(\theta_2^3-\theta_4^3\right)}{\theta_2+\theta_3+\theta_4}\left(1+4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)$$

$$c_2(\rho) = \frac{\pi^3\rho^{5/2}}{24\sqrt{2}}\frac{\theta_2\theta_3\theta_4\left(\theta_2^9+\theta_3^9+\theta_4^9\right)}{(\theta_2+\theta_3+\theta_4)^2}$$

$$\vdots$$

We have also used the following relations between derivatives of the elliptic functions:

$$\frac{\partial}{\partial\rho}\ln\theta_3 = \frac{\pi}{4}\theta_4^4+\frac{\partial}{\partial\rho}\ln\theta_2 \quad\text{and}\quad \frac{\partial}{\partial\rho}\ln\theta_4 = \frac{\pi}{4}\theta_3^4+\frac{\partial}{\partial\rho}\ln\theta_2.$$

Note that with the help of the identities

$$\frac{\partial}{\partial\rho}\ln\theta_2 = -\frac{1}{2}\theta_3^2E \quad\text{and}\quad \frac{\partial E}{\partial\rho} = \frac{\pi^2}{4}\theta_3^2\theta_4^4-\frac{\pi}{2}\theta_4^4E$$

one can express all derivatives of the elliptic functions in terms of the elliptic functions $\theta_2, \theta_3, \theta_4$ and the elliptic integral of the second kind E.

(ii) Similar expansion of the free energy of the dimer model is

$$f_{\text{bulk}} = -\frac{\gamma}{\pi}$$

$$f_0(\rho) = -\ln\frac{\theta_2^2+\theta_3^2+\theta_4^2}{2\eta^2}$$

$$f_1(\rho) = -\frac{\pi^3\rho^2}{90}\frac{\frac{7}{8}\left(\theta_2^{10}+\theta_3^{10}+\theta_4^{10}\right)+\theta_2^2\theta_3^2\theta_4^2\left[\theta_2^2\theta_4^2-\theta_2^2\theta_3^2-\theta_3^2\theta_4^2\right]}{\theta_2^2+\theta_3^2+\theta_4^2}$$

$$f_2(\rho) = -\frac{\pi^6\rho^4}{16\,200}\frac{\theta_3^2\left(\frac{7}{8}\theta_3^8+\theta_2^4\theta_4^4\right)^2+\theta_2^2\left(\frac{7}{8}\theta_2^8-\theta_3^4\theta_4^4\right)^2+\theta_4^2\left(\frac{7}{8}\theta_4^8-\theta_3^4\theta_2^4\right)^2}{\theta_2^2+\theta_3^2+\theta_4^2}$$

$$+\frac{\pi^6\rho^4}{16\,200}\left[\frac{\frac{7}{8}\left(\theta_2^{10}+\theta_3^{10}+\theta_4^{10}\right)+\theta_2^2\theta_3^2\theta_4^2\left[\theta_2^2\theta_4^2-\theta_2^2\theta_3^2-\theta_3^2\theta_4^2\right]}{\theta_2^2+\theta_3^2+\theta_4^2}\right]^2+\frac{\pi^6\rho^4}{756}$$

$$\times\frac{\theta_3^8\left(\theta_2^2-\theta_4^2\right)+\theta_4^8\left(\theta_2^2-\theta_3^2\right)+\frac{5}{8}\left[\theta_3^2\left(\theta_2^8-\theta_4^8\right)+\theta_4^2\left(\theta_2^8-\theta_3^8\right)\right]+\frac{5}{16}\left(2\theta_2^{10}-\theta_3^{10}-\theta_4^{10}\right)}{\theta_2^2+\theta_3^2+\theta_4^2}$$

$$+\frac{31\pi^5\rho^3}{12\,096}\frac{\theta_3^{10}\left(\theta_2^4-\theta_4^4\right)+\theta_2^{10}\left(\theta_3^4+\theta_4^4\right)-\theta_4^{10}\left(\theta_3^4+\theta_2^4\right)}{\theta_2^2+\theta_3^2+\theta_4^2}\left(1+4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)$$

$$-\frac{\pi^5\rho^3}{756}\frac{\theta_2^2\theta_3^2\theta_4^2\left[\theta_3^6\left(\theta_2^2-\theta_4^2\right)+\theta_2^6\left(\theta_3^2+\theta_4^2\right)-\theta_4^6\left(\theta_3^2+\theta_2^2\right)\right]}{\theta_2^2+\theta_3^2+\theta_4^2}\left(1+4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)$$

$$\vdots$$

(iii) Finally, the asymptotic expansion of the free energy and the internal energy of the Gaussian model after subtraction of the zero modes, $\ln \mu\sqrt{8S}$ and $-2/\mu^2$, respectively, can be written as

$$f_{\text{bulk}} = \frac{2\gamma}{\pi} - \ln\sqrt{2}$$

$$f_0(\rho) = \ln\sqrt{\rho}\,\eta^2$$

$$f_1(\rho) = \frac{\pi^3\rho^2}{180}\left(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4 - \theta_3^4\theta_4^4\right)$$

$$f_2(\rho) = \frac{\pi^6\rho^4}{1512}\theta_3^4\theta_4^4\left(\theta_2^8 - 2\theta_3^4\theta_4^4\right) + \frac{\pi^5\rho^3}{1512}\left(\theta_3^4+\theta_4^4\right)\left(\theta_3^4+\theta_2^4\right)\left(\theta_2^4-\theta_4^4\right)\left(1 + 4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)$$

$$\vdots$$

$$u_{\text{bulk}}(\rho) = -\frac{8}{\pi}\left(\ln\frac{\sqrt{S}}{\rho} + \ln\frac{\sqrt{2}}{\pi} + C_E - \frac{\pi}{4} - 2\ln\eta\right)$$

$$u_1(\rho) = -2 - \frac{4}{3}\rho\frac{\partial}{\partial\rho}\ln\eta - \frac{4}{3}\rho^2\frac{\partial^2}{\partial\rho^2}\ln\eta$$

$$u_3(\rho) = -\frac{\pi^5\rho^4}{270}\theta_3^4\theta_4^4\left(\theta_2^8 + 6\theta_3^4\theta_4^4\right) - \frac{\pi^4\rho^3}{54}\theta_3^4\theta_4^4\left(\theta_3^4+\theta_4^4\right)\left(1 + 4\rho\frac{\partial}{\partial\rho}\ln\theta_2\right)$$
$$- \frac{\pi^3\rho^2}{135}\left(\theta_2^8+\theta_3^4\theta_4^4\right)\left(\frac{43}{10} + 20\,\rho\frac{\partial}{\partial\rho}\,\ln\theta_2 + 32\,\rho^2\left(\frac{\partial}{\partial\rho}\ln\theta_2\right)^2\right.$$
$$\left. + 4\rho^2\frac{\partial^2}{\partial\rho^2}\ln\theta_2\right)$$

$$\vdots$$

5. Summary

In this paper, we have derived exact asymptotic expansion of the partition function with twisted boundary conditions at the critical point, equations (16) and (17). As an application of this result, we have obtained exact asymptotic expansion of the free energy, the internal energy and the specific heat for a class of free exactly solvable models of statistical mechanics on torus. Moreover, the partition function of the dimer model on the Klein bottle can also be expressed via the partition function with twisted boundary conditions, namely $Z_{\text{Klein}} = Z^2_{\frac{1}{4},\frac{1}{2}}(0)$ [19]. Exact asymptotic expansion of the latter can immediately be written down with the help of our general formula (16). An interesting question is whether this is also the case for other free models. This, however, is the problem for the future.

Very recently, the method described in this paper was used to obtain easily exact finite-size corrections for the square-lattice Ising model with Brascamp–Kunz boundary conditions up to very high orders [26], which suggests the usefulness of our method.

Acknowledgments

We are grateful to V B Priezzhev for critical remarks on the manuscript. This work was supported by the Russian Foundation for Basic Research through grant no 99-01-00882, by the Swiss National Science Foundation through grant no 7-SUPJ-62295 and by the National Science Council of the Republic of China (Taiwan) through grant no NSC 90-2112-M-001-074.

Appendix A. Euler–Maclaurin summation formula

Suppose that $F(x)$ together with its first $2m$ derivatives is continuous within the interval (a, b). Then the general Euler–Maclaurin summation formula states [20]

$$\sum_{n=0}^{N-1} F(a+nh+\alpha h) = \frac{1}{h}\int_a^b F(\tau)\,\mathrm{d}\tau + \sum_{k=1}^{p} \frac{h^{k-1}}{k!} B_k(\alpha)\left(F^{(k-1)}(b) - F^{(k-1)}(a)\right) - R_p(\alpha) \tag{A.1}$$

where $p \leqslant 2m$, $0 \leqslant \alpha \leqslant 1$, $h = (b-a)/N$ and remainder term $R_p(\alpha)$ is given by

$$R_p(\alpha) = \frac{h^p}{p!}\int_0^1 \hat{B}_p(\alpha - \tau)\left\{\sum_{n=0}^{N-1} F^{(p)}(a+nh+\tau h)\right\}\mathrm{d}\tau. \tag{A.2}$$

$\hat{B}_p(\alpha)$ are so-called periodic Bernoulli functions which are defined as follows:

$$\hat{B}_p(\alpha) = -\frac{p!}{(-2\pi \mathrm{i})^p}\sum_{n\neq 0}\frac{\mathrm{e}^{-2\pi \mathrm{i} n\alpha}}{n^p}. \tag{A.3}$$

These functions have singularities at the integer values of α and on the interval $\alpha \in [0, 1]$ they coincide with so-called Bernoulli polynomials:

$$B_1(\alpha) = \alpha - \tfrac{1}{2}$$
$$B_2(\alpha) = \alpha^2 - \alpha + \tfrac{1}{6}$$
$$B_3(\alpha) = \alpha^3 - \tfrac{3}{2}\alpha^2 + \tfrac{1}{2}\alpha$$
$$B_4(\alpha) = \alpha^4 - 2\alpha^3 + \alpha^2 - \tfrac{1}{30}.$$

The generating function for the Bernoulli polynomials is

$$\frac{\lambda\,\mathrm{e}^{\lambda\alpha}}{\mathrm{e}^{\lambda}-1} = 1 + \sum_{p=1}^{\infty}\frac{\lambda^p}{p!}B_p(\alpha).$$

Fourier transform of the generating function gives the important identity

$$\frac{\mathrm{e}^{2\pi \mathrm{i} z\alpha}}{\mathrm{e}^{2\pi \mathrm{i} z}-1} = -\sum_{n=0}^{\infty}\mathrm{e}^{2\pi \mathrm{i} z(n+\alpha)} = \frac{1}{2\pi \mathrm{i}}\sum_{n=-\infty}^{+\infty}\frac{\mathrm{e}^{-2\pi \mathrm{i} n\alpha}}{z+n}. \tag{A.4}$$

It is well known that Euler–Maclaurin summation formula is closely related to (generally speaking divergent) asymptotic series. For further discussion on the properties of these series, the interested reader is referred to the book by Hardy [21].

In this paper, we are mainly interested in sums of the form

$$\frac{1}{N}\sum_{n=0}^{N-1} f\left(\frac{\pi(n+\alpha)}{N}\right). \tag{A.5}$$

The asymptotic expansion of sum (A.5) in the limit $N \to \infty$ can be obtained from equation (A.1) by setting $a = 0$, $b = \pi$. If we assume that all the derivatives of $f(x)$ are integrable over the interval $(0, \pi)$, i.e. the integral in equation (A.2) is finite, we can formally extend the sum in equation (A.1) to $k = \infty$ and drop the reminder term $R_p(\alpha)$. In this case, we can write the asymptotic expansion of sum (A.5) as follows:

$$\frac{1}{N}\sum_{n=0}^{N-1} f\left(\frac{\pi(n+\alpha)}{N}\right) = \frac{1}{\pi}\int_0^{\pi} f(\tau)\,\mathrm{d}\tau + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{\pi}{N}\right)^k\frac{B_k(\alpha)}{k!}\left(f^{(k-1)}(\pi) - f^{(k-1)}(0)\right). \tag{A.6}$$

Note that we also use abbreviated notation for Bernoulli polynomials: $B_p(\alpha) = B_p^{\alpha}$.

Appendix B. Relation between moments and cumulants

Moments Z_k and cumulants F_k which enter the expansion of exponent,

$$\exp\left\{\sum_{k=1}^{\infty}\frac{x^k}{k!}F_k\right\} = 1+\sum_{k=1}^{\infty}\frac{x^k}{k!}Z_k$$

are related to each other as [22]

$$\begin{aligned}
Z_1 &= F_1\\
Z_2 &= F_2 + F_1^2\\
Z_3 &= F_3 + 3F_1F_2 + F_1^3\\
Z_4 &= F_4 + 4F_1F_3 + 3F_2^2 + 6F_1^2F_2 + F_1^4\\
&\vdots\\
Z_k &= \sum_{r=1}^{k}\sum\left(\frac{F_{k_1}}{k_1!}\right)^{i_1}\cdots\left(\frac{F_{k_r}}{k_r!}\right)^{i_r}\frac{k!}{i_1!\ldots i_r!}
\end{aligned}$$

where summation is over all positive numbers $\{i_1,\ldots,i_r\}$ and different positive numbers $\{k_1,\ldots,k_r\}$ such that $k_1i_1+\cdots+k_ri_r = k$.

Appendix C. Elliptic theta functions

We adopt the following definition of the elliptic θ-functions:

$$\begin{aligned}
\theta_{\alpha,\beta}(z,\tau) &= \sum_{n\in Z}\exp\left\{\pi\mathrm{i}\tau\left(n+\tfrac{1}{2}-\alpha\right)^2+2\pi\mathrm{i}\left(n+\tfrac{1}{2}-\alpha\right)\left(z+\tfrac{1}{2}-\beta\right)\right\}\\
&= \eta(\tau)\exp\left\{\pi\mathrm{i}\tau\left(\alpha^2-\alpha+\tfrac{1}{6}\right)+2\pi\mathrm{i}\left(\tfrac{1}{2}-\alpha\right)\left(z+\tfrac{1}{2}-\beta\right)\right\}\\
&\quad\times\prod_{n=0}^{\infty}[1-\exp\{2\pi\mathrm{i}\tau(n+\alpha)-2\pi\mathrm{i}(z-\beta)\}]\\
&\quad\times[1-\exp\{2\pi\mathrm{i}\tau(n+1-\alpha)+2\pi\mathrm{i}(z-\beta)\}].
\end{aligned}$$

These should be compared with the notations of Mumford [23].

The elliptic θ-functions satisfy the heat equation

$$\frac{\partial}{\partial\tau}\theta_{\alpha,\beta}(z,\tau) = \frac{1}{4\pi\mathrm{i}}\frac{\partial^2}{\partial z^2}\theta_{\alpha,\beta}(z,\tau). \tag{C.1}$$

Relations to standard notations are

$$\begin{aligned}
\theta_{0,0}(z,\tau) &= \theta_1(z,\tau)\\
\theta_{0,\frac{1}{2}}(z,\tau) &= \theta_2(z,\tau)\\
\theta_{\frac{1}{2},0}(z,\tau) &= \theta_4(z,\tau)\\
\theta_{\frac{1}{2},\frac{1}{2}}(z,\tau) &= \theta_3(z,\tau).
\end{aligned}$$

The Dedekind η-function is usually defined as

$$\eta(\tau) = \mathrm{e}^{\pi\mathrm{i}\tau/12}\prod_{n=1}^{\infty}[1-\mathrm{e}^{2\pi\mathrm{i}\tau n}].$$

Considering the functions $\theta_{\alpha,\beta}(\tau) = \theta_{\alpha,\beta}(0,\tau)$ and $\eta(\tau)$ of pure imaginary aspect ratio, $\tau = \mathrm{i}\rho$, we obtain the identity

$$\ln\left|\frac{\theta_{\alpha,\beta}(\mathrm{i}\rho)}{\eta(\mathrm{i}\rho)}\right| + \pi\rho B_2^\alpha = \sum_{n=0}^{\infty} \ln|1-\exp\{-2\pi\rho(n+\alpha) - 2\pi\mathrm{i}\beta\}| + \sum_{n=0}^{\infty} \ln|1-\exp\{-2\pi\rho(n+1-\alpha) - 2\pi\mathrm{i}\beta\}|. \tag{C.2}$$

Taking the derivative of equation (C.2) with respect to ρ we can obtain the following useful identity:

$$\mathtt{Re}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[(n+\alpha)\exp\{-2\pi m[\rho(n+\alpha)+\mathrm{i}\beta]\} + (n+1-\alpha)\exp\{-2\pi m[\rho(n+1-\alpha)+\mathrm{i}\beta]\}] = \frac{B_2^\alpha}{2} + \frac{1}{2\pi}\frac{\partial}{\partial\rho}\ln\left|\frac{\theta_{\alpha,\beta}(\mathrm{i}\rho)}{\eta(\mathrm{i}\rho)}\right|. \tag{C.3}$$

Taking the second derivative of $\ln|\theta_{\alpha,\beta}(z,\mathrm{i}\rho)|$ with respect to z at $z = 0$ and using the heat equation (C.1) we obtain

$$\frac{\partial}{\partial\rho}\left(\mathtt{Re}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{n+\alpha}\exp\{-2\pi m[\rho(n+\alpha)+\mathrm{i}\beta]\} + \frac{1}{n+1-\alpha}\exp\{-2\pi m[\rho(n+1-\alpha)+\mathrm{i}\beta]\}\right\}\right) = -2\frac{\partial}{\partial\rho}\ln\left|\theta_{\alpha,\beta}(\mathrm{i}\rho)\right| + \frac{1}{2\pi}\left(\frac{\partial}{\partial z}\ln\left|\theta_{\alpha,\beta}(\mathrm{i}\rho)\right|\right)^2. \tag{C.4}$$

Appendix D. Kronecker's double series

Kronecker's double series can be defined as [24]

$$K_p^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi\mathrm{i})^p}\sum_{\substack{m,n\in Z\\(m,n)\neq(0,0)}}\frac{\mathrm{e}^{-2\pi\mathrm{i}(n\alpha+m\beta)}}{(n+\tau m)^p}.$$

In this form, however, they cannot be directly applied to our analysis. We need to cast them in a different form. To this end, let us separate from the double series a subseries with $m = 0$:

$$K_p^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi\mathrm{i})^p}\sum_{n\neq0}\frac{\mathrm{e}^{-2\pi\mathrm{i}n\alpha}}{n^p} - \frac{p!}{(-2\pi\mathrm{i})^p}\sum_{m\neq0}\sum_{n\in Z}\frac{\mathrm{e}^{-2\pi\mathrm{i}(n\alpha+m\beta)}}{(n+\tau m)^p}.$$

Here the first sum gives nothing but Fourier representation of Bernoulli polynomials (A.3). The second sum can be rearranged with the help of the identity

$$\frac{p!}{(-2\pi\mathrm{i})^p}\sum_{n\in Z}\frac{\mathrm{e}^{-2\pi\mathrm{i}n\alpha}}{(z+n)^p} = p\sum_{n=0}^{\infty}(n+\alpha)^{p-1}\,\mathrm{e}^{2\pi\mathrm{i}z(n+\alpha)}$$

which can easily be derived from equation (A.4) differentiating it p times. The final result of our resummation of the double Kronecker sum is

$$K_p^{\alpha,\beta}(\tau) = B_p^\alpha - p\sum_{m\neq0}\sum_{n=0}^{\infty}(n+\alpha)^{p-1}\exp\{2\pi\mathrm{i}m(\tau(n+\alpha)-\beta)\}.$$

Considering real part of the Kronecker sums with pure imaginary aspect ratio, $\tau = \mathrm{i}\rho$, we can further rearrange this expression to get summation only over positive $m \geqslant 1$:

$$B_{2p}^{\alpha} - \mathtt{Re}\, K_{2p}^{\alpha,\beta}(\mathrm{i}\rho) = \frac{(2p)!}{(-4\pi^2)^p}\mathtt{Re}\sum_{m=1}^{\infty}\sum_{n\in Z}\frac{\mathrm{e}^{-2\pi\mathrm{i}(n\alpha+m\beta)} + \mathrm{e}^{-2\pi\mathrm{i}(n(1-\alpha)-m\beta)}}{(n+\mathrm{i}\rho m)^{2p}}$$

$$= 2p\,\mathtt{Re}\left\{\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(n+\alpha)^{2p-1}\exp\{-2\pi m(\rho(n+\alpha)+i\beta)\}\right.$$

$$\left. + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(n+1-\alpha)^{2p-1}\exp\{-2\pi m(\rho(n+1-\alpha)+\mathrm{i}\beta)\}\right\}. \qquad \text{(D.1)}$$

Appendix E. Asymptotic expansion of the digamma function

Let us start with well-known expansion of the digamma function $\psi(N)$ [14]:

$$\psi(x) = \ln x - \frac{1}{2x} - \sum_{p=1}^{\infty}\frac{B_{2p}}{2p}\frac{1}{x^{2p}}$$

$$= \ln x - \sum_{p=1}^{\infty}(-1)^p\frac{B_p}{p}\frac{1}{x^p}. \qquad \text{(E.1)}$$

Plugging in the above expansion $x = N + \alpha$ and expanding the resulting factors $\ln(1+\alpha/N)$, $(1+\alpha/N)^{-p}$ in powers of N^{-1} we obtain

$$\psi(N+\alpha) = \ln N - \sum_{p=1}^{\infty}(-1)^p\frac{\alpha^p}{pN^p} - \sum_{p=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{k+p}B_p\frac{(p+k-1)!}{k!\,p!}\frac{\alpha^k}{N^{p+k}}$$

$$= \ln N - \sum_{p=1}^{\infty}(-1)^p\frac{\alpha^p}{pN^p} - \sum_{l=1}^{\infty}\sum_{p=1}^{l}(-1)^l B_p\frac{(l-1)!}{(l-p)!\,p!}\frac{\alpha^{l-p}}{N^l}$$

$$= \ln N - \sum_{l=1}^{\infty}\sum_{p=0}^{l}(-1)^l B_p\frac{(l-1)!}{(l-p)!\,p!}\frac{\alpha^{l-p}}{N^l}. \qquad \text{(E.2)}$$

Using the relation between Bernoulli polynomials B_p^{α} and Bernoulli numbers B_p,

$$B_l^{\alpha} = \sum_{p=0}^{l}B_p\frac{l!}{(l-p)!\,p!}\alpha^{l-p} \qquad \text{(E.3)}$$

we finally obtain equation (22):

$$\psi(N+\alpha) = \ln N - \sum_{p=1}^{\infty}(-1)^p\frac{B_p^{\alpha}}{p}\frac{1}{N^p}. \qquad \text{(E.4)}$$

Appendix F. Reduction of Kronecker's double series to theta functions

Let us consider Laurent expansion of the Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)}\left[\frac{1}{(z-n-\tau m)^2} - \frac{1}{(n+\tau m)^2}\right]$$

$$= \frac{1}{z^2} + \sum_{p=2}^{\infty}a_p(\tau)z^{2p-2}.$$

The coefficients $a_p(\tau)$ of the expansion can all be written in terms of the elliptic θ-functions with the help of the recursion relation [25]

$$a_p = \frac{3}{(p-3)(2p+1)}(a_2 a_{p-2} + a_3 a_{p-3} + \cdots + a_{p-2} a_2)$$

where first terms of the sequence are

$$\begin{aligned}
a_2 &= \frac{\pi^4}{15}\left(\theta_2^4\theta_3^4 - \theta_2^4\theta_4^4 + \theta_3^4\theta_4^4\right)\\
a_3 &= \frac{\pi^6}{189}\left(\theta_2^4+\theta_3^4\right)\left(\theta_4^4-\theta_2^4\right)\left(\theta_3^4+\theta_4^4\right)\\
a_4 &= \frac{1}{3}a_2^2\\
a_5 &= \frac{3}{11}(a_2a_3)\\
a_6 &= \frac{1}{39}\left(2a_2^3+3a_3^2\right)\\
&\vdots
\end{aligned}$$

Kronecker functions $K_{2p}^{0,0}(\tau)$ are related directly to the coefficients $a_p(\tau)$:

$$K_{2p}^{0,0}(\tau) = -\frac{(2p)!}{(-4\pi^2)^p}\frac{a_p(\tau)}{(2p-1)}.$$

Kronecker functions $K_{2p}^{\alpha,\beta}(\tau)$ with α and β taking values 0 or $1/2$ can in their turn be related to the function $K_{2p}^{0,0}(\tau)$ by means of simple resummation of Kronecker's double series

$$\begin{aligned}
K_p^{0,\frac{1}{2}}(\tau) &= 2K_p^{0,0}(2\tau) - K_p^{0,0}(\tau)\\
K_p^{\frac{1}{2},0}(\tau) &= 2^{1-p}K_p^{0,0}(\tau/2) - K_p^{0,0}(\tau)\\
K_p^{\frac{1}{2},\frac{1}{2}}(\tau) &= (1+2^{2-p})K_p^{0,0}(\tau) - 2^{1-p}K_p^{0,0}(\tau/2) - 2K_p^{0,0}(2\tau).
\end{aligned}$$

Thus, Kronecker functions $K_{2p}^{\alpha,\beta}(\tau)$ with α and β taking values 0 or $1/2$ can all be expressed in terms of the elliptic θ-functions only. For practical calculations, the following identities are also helpful:

$$\begin{aligned}
2\theta_2^2(2\tau) &= \theta_3^2 - \theta_4^2 &\qquad \theta_2^2(\tau/2) &= 2\theta_2\theta_3\\
2\theta_3^2(2\tau) &= \theta_3^2 + \theta_4^2 &\qquad \theta_3^2(\tau/2) &= \theta_2^2 + \theta_3^2\\
2\theta_4^2(2\tau) &= 2\theta_3\theta_4 &\qquad \theta_4^2(\tau/2) &= \theta_3^2 - \theta_2^2.
\end{aligned}$$

From the general formulae above, we can easily write down all Kronecker functions that have appeared in our asymptotic expansions:

$$\begin{aligned}
K_4^{0,0}(\tau) &= \tfrac{1}{30}\left(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4 - \theta_3^4\theta_4^4\right)\\
K_4^{0,\frac{1}{2}}(\tau) &= \tfrac{1}{30}\left(\tfrac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4\right)\\
K_4^{\frac{1}{2},0}(\tau) &= \tfrac{1}{30}\left(\tfrac{7}{8}\theta_4^8 - \theta_2^4\theta_3^4\right)\\
K_4^{\frac{1}{2},\frac{1}{2}}(\tau) &= \tfrac{1}{30}\left(\tfrac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4\right)\\
K_6^{0,0}(\tau) &= \tfrac{1}{84}\left(\theta_2^4+\theta_3^4\right)\left(\theta_4^4-\theta_2^4\right)\left(\theta_3^4+\theta_4^4\right)
\end{aligned}$$

$$K_6^{0,\frac{1}{2}}(\tau) = \frac{1}{84}\left(\theta_3^4 + \theta_4^4\right)\left(\frac{31}{16}\theta_2^8 + \theta_3^4\theta_4^4\right)$$
$$K_6^{\frac{1}{2},0}(\tau) = -\frac{1}{84}\left(\theta_2^4 + \theta_3^4\right)\left(\frac{31}{16}\theta_4^8 + \theta_2^4\theta_3^4\right)$$
$$K_6^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{1}{84}\left(\theta_2^4 - \theta_4^4\right)\left(\frac{31}{16}\theta_3^8 - \theta_4^4\theta_2^4\right)$$
$$\vdots$$

Note that when $\rho \to \infty$ we have limits $\theta_2 \to 0$, $\theta_4 \to 1$, $\theta_3 \to 1$ and Kronecker's function reduces to the Bernoulli polynomials.

References

[1] Privman V 1990 *Finite Size Scaling and Numerical Simulation of Statistical Systems* ed V Privman (Singapore: World Scientific) see also references therein

[2] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

[3] McCoy B and Wu T T 1973 *The Two-Dimensional Ising Model* (Cambridge: Harvard University Press)

[4] Cardy J L 1990 *Conformal Invariance and Statistical Mechanics, Les Houches Session XLIX: Fields, Strings and Critical Phenomena* ed E Brezin and J Zinn-Justin (Amsterdam: North-Holland)

[5] Izmailian N Sh and Hu C K 2001 *Phys. Rev. Lett.* **86** 5160

[6] Ferdinand A E and Fisher M E 1969 *Phys. Rev.* **185** 832

[7] Onsager L 1944 *Phys. Rev.* **65** 117

[8] Hu C K, Chen J A, Izmailian N Sh and Kleban P 1999 *Phys. Rev.* E **60** 6491 (*Preprint* cond-mat/9905203)

[9] Izmailian N Sh and Hu C K 2002 *Phys. Rev.* E **65** 036103 (*Preprint* cond-mat/0009024v2)

[10] Salas J 2001 *J. Phys. A: Math. Gen.* **34** 1311 (*Preprint* cond-mat/0009054v3)

[11] Salas J and Sokal A D 1997 *J. Stat. Phys.* **88** 567

[12] Christe P and Henkel M 1993 *Introduction to Conformal Invariance and its Application to Critical Phenomena* (Berlin: Springer)

[13] Kaufmann B 1949 *Phys. Rev.* **76** 1232

[14] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)

[15] Kasteleyn P W 1961 *Physica* **27** 1209

[16] Fisher M E 1961 *Phys. Rev.* **124** 1664

[17] Ferdinand A E 1967 *J. Math. Phys.* **8** 2332

[18] Sato M, Miwa T and Jimbo M 1980 *Holonomic Quantum Fields V* vol 16 (RIMS, Kyoto University) p 531

[19] Lu W T and Wu F Y 1999 *Phys. Lett.* A **259** 108

[20] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover) p 806

[21] Hardy G H 1949 *Divergent Series* (Oxford: Clarendon)

[22] Prohorov Yu V and Rozanov Yu A 1969 *Probability Theory* (New York: Springer)

[23] Mumford D 1983 *Tata Lectures on Theta* (Basel: Birkhäuser)

[24] Weil A 1976 *Elliptic Functions According to Eisenshtein and Kronecker* (Berlin: Springer)

[25] Korn G A and Korn T M 1968 *Mathematical Handbook* (New York: McGraw-Hill)

[26] Izmailian N Sh., Oganesyan K B and Hu C-K 2002 *Phys. Rev. E* **65** 056132 (*Preprint* cond-mat/0202282)